Exercise 7

A First Principles

1 Solution

(a) $$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{(x+\delta x-5)-(x-5)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\delta x}{\delta x} \\ &= \lim_{\delta x \to 0} (1) \\ &= 1 \end{aligned}$$

(b) $$\begin{aligned} f'(x) &= \lim_{\delta x \to 0} \frac{f(x+\delta x)-f(x)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\left((x+\delta x)^2-2\right)-\left(x^2-2\right)}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{x^2+2(\delta x)(x)+(\delta x)^2-2-x^2+2}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{2(\delta x)(x)+(\delta x)^2}{\delta x} \\ &= \lim_{\delta x \to 0} \frac{\delta x\left[2x+(\delta x)\right]}{\delta x} \\ &= \lim_{\delta x \to 0} \left[2x+(\delta x)\right] \\ &= 2x \end{aligned}$$

(c) $$f'(x) = \lim_{\delta x \to 0} \frac{f(x+\delta x) - f(x)}{\delta x}$$

$$= \lim_{\delta x \to 0} \left(\frac{\tan(x+\delta x) - \tan x}{\delta x} \right)$$

$$= \lim_{\delta x \to 0} \left(\frac{\dfrac{\tan x + \tan \delta x}{1 - \tan x \tan \delta x} - \tan x}{\delta x} \right)$$

$$= \lim_{\delta x \to 0} \left(\frac{\tan \delta x + \tan^2 x \tan \delta x}{\delta x(1 - \tan x \tan \delta x)} \right)$$

$$= \lim_{\delta x \to 0} \frac{(1 + \tan^2 x)}{\delta x \left(\dfrac{1}{\tan \delta x} - \tan x \right)}$$

$$= \lim_{\delta x \to 0} \frac{(1 + \tan^2 x)}{\delta x \left(\dfrac{\cos \delta x}{\sin \delta x} - \tan x \right)}$$

$$= \lim_{\delta x \to 0} \frac{(\sec^2 x)}{\delta x \left(\dfrac{\cos \delta x - \sin \delta x \tan x}{\sin \delta x} \right)}$$

$$= \lim_{\delta x \to 0} \frac{\sin \delta x (\sec^2 x)}{\delta x (\cos \delta x - \sin \delta x \tan x)}$$

$$= \lim_{\delta x \to 0} \frac{\sin \delta x}{\delta x} \lim_{\delta x \to 0} \frac{(1 + \tan^2 x)}{(\cos \delta x - \sin \delta x \tan x)}$$

$$= (1) \frac{(\sec^2 x)}{(1 - 0)}$$

$$= \sec^2 x$$

Exercise 7

B Derivatives of Algebraic Functions

2 Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sqrt{x}+\frac{1}{2\sqrt{x}}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}\right)$

$=\frac{1}{2}x^{-\frac{1}{2}}-\frac{1}{4}x^{-\frac{3}{2}}$

$=\frac{1}{2\sqrt{x}}+\left(-\frac{1}{4\sqrt{x^3}}\right)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3x^2+x-1}{\sqrt{x}}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{3x^2}{\sqrt{x}}+\frac{x}{\sqrt{x}}-\frac{1}{\sqrt{x}}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(3x^{\frac{3}{2}}+x^{\frac{1}{2}}-x^{-\frac{1}{2}}\right)$

$=\frac{9}{2}x^{\frac{1}{2}}+\frac{1}{2}x^{-\frac{1}{2}}+\frac{1}{2}x^{-\frac{3}{2}}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{(x+1)(2x-3)}{x}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x^2-x-3}{x}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2x^2}{x}-\frac{x}{x}-\frac{3}{x}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(2x-1-3x^{-1}\right)$

$=2+3x^{-2}$

$=2+\frac{3}{x^2}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(-\frac{a}{3}+\frac{b}{5}x^7\right)$

$=\frac{7b}{5}x^6$

3 Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x^2 - x + 1})$

$= \frac{\mathrm{d}}{\mathrm{d}x}(x^2 - x + 1)^{\frac{1}{2}}$

$= \frac{1}{2}\left(x^2 - x + 1\right)^{-\frac{1}{2}}\left(2x - 1\right)$

$= \frac{2x - 1}{2\sqrt{x^2 - x + 1}}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(3x - 2)^4$

$= (4)\left(3x - 2\right)^3 (3)$

$= 12\left(3x - 2\right)^3$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(5x^2 + \frac{1}{3}x^{-2}\right)^2$

$= 2\left(5x^2 + \frac{1}{3}x^{-2}\right)\left(10x - \frac{2}{3}x^{-3}\right)$

$= 2\left(5x^2 + \frac{1}{3x^2}\right)\left(10x - \frac{2}{3x^3}\right)$

Alternative Method

$\frac{\mathrm{d}}{\mathrm{d}x}\left(5x^2 + \frac{1}{3x^2}\right)^2$

$= \frac{\mathrm{d}}{\mathrm{d}x}\left(5x^2 + \frac{1}{3}x^{-2}\right)^2$

$= \frac{\mathrm{d}}{\mathrm{d}x}\left(25x^4 + \frac{10}{3} + \frac{1}{9}x^{-4}\right)$

$= 100x^3 - \frac{4}{9}x^{-5}$

$= 100x^3 - \frac{4}{9x^5}$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{1}{\sqrt{ax^2-b^2}}\right)$

$$= \frac{\mathrm{d}}{\mathrm{d}x}(ax^2-b^2)^{-\frac{1}{2}}$$

$$= -\frac{1}{2}(ax^2-b^2)^{-\frac{3}{2}}(2ax)$$

$$= -\frac{ax}{(ax^2-b^2)^{\frac{3}{2}}}$$

4 Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(x+6)^7(x-9)^8$

$$= (x+6)^7 \frac{\mathrm{d}}{\mathrm{d}x}(x-9)^8 + (x-9)^8 \frac{\mathrm{d}}{\mathrm{d}x}(x+6)^7$$

$$= (x+6)^7\, 8(x-9)^7 + (x-9)^8\, 7(x+6)^6$$

$$= (x+6)^6 (x-9)^7 \left[8(x+6)+7(x-9)\right]$$

$$= (x+6)^6 (x-9)^7 \left[8x+48+7x-63\right]$$

$$= (x+6)^6 (x-9)^7 (15x-15)$$

$$= 15(x+6)^6 (x-9)^7 (x-1)$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(2x+1)(3-x^2)^{\frac{1}{2}}$

$$(2x+1)\frac{\mathrm{d}}{\mathrm{d}x}(3-x^2)^{\frac{1}{2}} + \sqrt{3-x^2}\,\frac{\mathrm{d}}{\mathrm{d}x}(2x+1)$$

$$= (2x+1)\left[\frac{1}{2}(3-x^2)^{-\frac{1}{2}}(-2x)\right] + \sqrt{3-x^2}\times(2)$$

$$= (2x+1)\left(3-x^2\right)^{-\frac{1}{2}}(-x) + 2\sqrt{3-x^2}$$

$$= \left(3-x^2\right)^{-\frac{1}{2}}\left[-x(2x+1)+2(3-x^2)\right]$$

$$= \left(3-x^2\right)^{-\frac{1}{2}}\left[-2x^2-x+6-2x^2)\right]$$

$$= \frac{-4x^2-x+6}{\sqrt{3-x^2}}$$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4(x+1)^2\right)$

$$= x^4\, 2(x+1) + (x+1)^2\, 4x^3$$

$$= 2x^3(x+1)[x+2(x+1)]$$

$$= 2x^3(x+1)(3x+2)$$

Alternative Method

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^4(x+1)^2\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^4\left(x^2+2x+1\right)\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(x^6+2x^5+x^4\right)$$

$$= 6x^5+10x^4+4x^3$$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left[(a^2+3x^2)(b^2+x^2)^{\frac{1}{2}}\right]$

$$= (a^2+3x^2)\frac{1}{2}(b^2+x^2)^{-\frac{1}{2}}(2x)+(b^2+x^2)^{\frac{1}{2}}(6x)$$

$$= x(b^2+x^2)^{-\frac{1}{2}}(a^2+3x^2)+6x(b^2+x^2)^{\frac{1}{2}}$$

$$= x(b^2+x^2)^{-\frac{1}{2}}\left[(a^2+3x^2)+6(b^2+x^2)\right]$$

$$= \frac{x}{\sqrt{b^2+x^2}}(a^2+3x^2+6b^2+6x^2)$$

$$= \frac{x}{\sqrt{b^2+x^2}}(9x^2+a^2+6b^2)$$

5 Solution

(a) Let $y=\dfrac{x}{2x+5}$

$$\frac{dy}{dx}=\frac{(2x+5)\frac{d}{dx}(x)-(x)\frac{d}{dx}(2x+5)}{(2x+5)^2}$$
$$=\frac{(2x+5)(1)-(x)(2)}{(2x+5)^2}$$
$$=\frac{2x+5-2x}{(2x+5)^2}$$
$$=\frac{5}{(2x+5)^2}$$

(b) $\dfrac{d}{dx}\left(\dfrac{x^2+x-1}{1-2x}\right)$

$$=\frac{(1-2x)(2x+1)-(x^2+x-1)(-2)}{(1-2x)^2}$$
$$=\frac{2x+1-4x^2-2x+2x^2+2x-2}{(1-2x)^2}$$
$$=\frac{-2x^2+2x-1}{(1-2x)^2}$$

(c) $\dfrac{d}{dx}\left(\dfrac{\sqrt{x}}{3+x}\right)$

$$=\frac{(3+x)\frac{1}{2}x^{-\frac{1}{2}}-\sqrt{x}}{(3+x)^2}$$
$$=\frac{\frac{1}{2}x^{-\frac{1}{2}}\left[(3+x)-2(x)\right]}{(3+x)^2}$$
$$=\frac{3+x-2x}{2\sqrt{x}(3+x)^2}$$
$$=\frac{3-x}{2\sqrt{x}(3+x)^2}$$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{x+1}{\sqrt{1-3x}}\right)$

$$=\frac{\sqrt{1-3x}(1)-(x+1)\dfrac{1}{2}(1-3x)^{-\frac{1}{2}}(-3)}{(1-3x)}$$

$$=\frac{\sqrt{1-3x}+\dfrac{3}{2}(x+1)(1-3x)^{-\frac{1}{2}}}{(1-3x)}$$

$$=\frac{\dfrac{1}{2}(1-3x)^{-\frac{1}{2}}\left[2(1-3x)+3(x+1)\right]}{(1-3x)}$$

$$=\frac{2-6x+3x+3}{2(1-3x)^{\frac{3}{2}}}$$

$$=\frac{5-3x}{2(1-3x)^{\frac{3}{2}}}$$

Exercise 7

C Derivatives of Trigonometric Functions

6
Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\operatorname{cosec}\left(2x+3\right)\right)$

$= -\operatorname{cosec}\left(2x+3\right)\cot\left(2x+3\right)(2)$

$= -2\operatorname{cosec}\left(2x+3\right)\cot\left(2x+3\right)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot(1-2x^2)\right)$

$= -\operatorname{cosec}^2(1-2x^2)\times(-4x)$

$= 4x\operatorname{cosec}^2(1-2x^2)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sec 3x-\cot 5x\right)$

$= 3\sec 3x\tan 3x + 5\operatorname{cosec}^2 5x$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan\frac{1}{2}x-3\cos x^2\right)$

$= \frac{1}{2}\sec^2\frac{1}{2}x-3(-\sin x^2)(2x)$

$= \frac{1}{2}\sec^2\frac{1}{2}x+6x\sin x^2$

6
Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan^2 3x)$

$= \frac{\mathrm{d}}{\mathrm{d}x}(\tan 3x)^2$

$= 2(\tan 3x)(\sec^2 3x)(3)$

$= 6\tan 3x \sec^2 3x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{2}{\sin 4x}\right)$

$\frac{\mathrm{d}}{\mathrm{d}x}\left(2(\sin 4x)^{-1}\right)$

$= -2(\sin 4x)^{-2}\cos 4x \times (4)$

$= -\frac{8\cos 4x}{\sin^2 4x}$

$= -8\,\mathrm{cosec}\,4x \cot 4x$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{1+\cos 4x}\right)$

$= \frac{\mathrm{d}}{\mathrm{d}x}\left(1+\cos 4x\right)^{-1}$

$= -\left(1+\cos 4x\right)^{-2}\left(-4\sin 4x\right)$

$= -\frac{1}{\left(1+\cos 4x\right)^2}\left(-4\sin 4x\right)$

$= \frac{4\sin 4x}{\left(1+\cos 4x\right)^2}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(2+\sin x)^2$

$= 2(2+\sin x)\left(\cos x\right)$

$= 2\cos x(2+\sin x)$

8
Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2\cos(x^3))$

$= (2x)\left[\cos(x^3)\right] + x^2 \cdot \left[-\sin(x^3)\right] \cdot (3x^2)$

$= 2x\cos(x^3) - 3x^4\sin(x^3)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\cot\frac{x}{2}\operatorname{cosec}\frac{x}{2}\right)$

$= \left(-\operatorname{cosec}^2\frac{x}{2}\right)\left(\frac{1}{2}\right)\operatorname{cosec}\frac{x}{2} + \left(\cot\frac{x}{2}\right)\left(-\operatorname{cosec}\frac{x}{2}\cot\frac{x}{2}\right)\left(\frac{1}{2}\right)$

$= -\frac{1}{2}\operatorname{cosec}^3\frac{x}{2} - \frac{1}{2}\operatorname{cosec}\frac{x}{2}\cot^2\frac{x}{2}$

$= -\frac{1}{2}\operatorname{cosec}\frac{x}{2}\left(\operatorname{cosec}^2\frac{x}{2} + \cot^2\frac{x}{2}\right)$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\tan x^2}{x}\right)$

$= \frac{x(2x)\sec^2 x^2 - (\tan x^2)(1)}{x^2}$

$= \frac{2x^2\sec^2 x^2 - \tan x^2}{x^2}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\sec x}{1+\tan x}\right)$

$= \frac{(1+\tan x)(\sec x\tan x) - (\sec x)(\sec^2 x)}{(1+\tan x)^2}$

$= \frac{(1+\tan x)(\sec x\tan x) - \sec^3 x}{(1+\tan x)^2}$

$= \frac{\sec x(\tan x + \tan^2 x - \sec^2 x)}{(1+\tan x)^2}$

$= \frac{\sec x(\tan x + \sec^2 x - 1 - \sec^2 x)}{(1+\tan x)^2}$

$= \frac{\sec x(\tan x - 1)}{(1+\tan x)^2}$

Exercise 7

D Derivatives of Exponential Functions

9 Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(\mathrm{e}^{1+\sin x})$

$= \cos x \ \mathrm{e}^{1+\sin x}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 - 4\mathrm{e}^{-2x}\right)$

$= 3x^2 + 8\mathrm{e}^{-2x}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x}\left(\mathrm{e}^{5x} - \mathrm{e}^{-3x}\right)\right)$

$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{7x} - \mathrm{e}^{-x}\right)$

$= 7\mathrm{e}^{7x} + \mathrm{e}^{-x}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}(5^{4x} + 3^{\cot x})$

$= \frac{\mathrm{d}}{\mathrm{d}x}(5^{4x}) + \frac{\mathrm{d}}{\mathrm{d}x}(3^{\cot x})$

$= 4\ln 5(5^{4x}) - \ln 3 \operatorname{cosec}^2 x(3^{\cot x})$

10 Solution

(a) $\dfrac{d}{dx}\left(\dfrac{3e^{3x}-4e^{x}}{e^{-x}}\right)$

$=\dfrac{d}{dx}(e^{-x}(3e^{3x}-4e^{x}))$

$=\dfrac{d}{dx}(3e^{4x}-4e^{2x})$

$=12e^{4x}-8e^{2x}$

(b) $\dfrac{d}{dx}\left(1+\dfrac{1}{e^{2x}}\right)^{3}$

$=\dfrac{d}{dx}(1+e^{-2x})^{3}$

$=3(1+e^{-2x})^{2}(-2e^{-2x})$

$=-6e^{-2x}(1+e^{-2x})^{2}$

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Solution

(a) $\dfrac{d}{dx}(x^2e^x)$

$= x^2(e^x) + e^x(2x)$

$= x^2e^x + 2xe^x$

(b) $\dfrac{d}{dx}(e^{3x}\sec 4x)$

$= e^{3x}4\tan 4x\sec 4x + \sec 4x(3e^{3x})$

$= e^{3x}\sec 4x\left(4\tan 4x + 3\right)$

(c) $\dfrac{d}{dx}\left(\dfrac{e^x}{\sin^2 2x}\right)$

$= \dfrac{(\sin^2 2x)e^x - e^x 2(\sin 2x)\left(2\cos 2x\right)}{(\sin^2 2x)^2}$

$= \dfrac{(\sin^2 2x)e^x - e^x 2\left(2\sin 2x\cos 2x\right)}{(\sin^2 2x)^2}$

$= \dfrac{(\sin^2 2x)e^x - e^x 2\left(\cos 4x\right)}{\sin^4 2x}$

$= \dfrac{e^x(\sin^2 2x - 2\cos 4x)}{\sin^4 2x}$

(d) $\dfrac{d}{dx}\left(\dfrac{e^{-2x}-1}{(1-x)^2}\right)$

$= \dfrac{(1-x)^2(-2e^{-2x}) - (e^{-2x}-1)2(1-x)(-1)}{(1-x)^4}$

$= \dfrac{(1-x)\left[(1-x)(-2e^{-2x}) + 2(e^{-2x}-1)\right]}{(1-x)^4}$

$= \dfrac{(1-x)(-2e^{-2x}) + 2(e^{-2x}-1)}{(1-x)^3}$

$= \dfrac{-2e^{-2x} + 2xe^{-2x} + 2e^{-2x} - 2}{(1-x)^3}$

$= \dfrac{2xe^{-2x} - 2}{(1-x)^3}$

Exercise 7

E Derivatives of Logarithm Functions

12 Solution

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}[\ln(1+x^3)]$

$$= \frac{3x^2}{1+x^3}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(\log_x 2)$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln 2}{\ln x}\right)$$

$$= (\ln 2)\frac{\mathrm{d}}{\mathrm{d}x}((\ln x)^{-1})$$

$$= (\ln 2)\left[-(\ln x)^{-2}\frac{\mathrm{d}}{\mathrm{d}x}(\ln x)\right]$$

$$= -(\ln 2)\frac{1}{(\ln x)^2}\frac{1}{x}$$

$$= -\frac{\ln 2}{x(\ln x)^2}$$

(c) $\dfrac{\mathrm{d}}{\mathrm{d}x}\big(x\ln(\operatorname{cosec} 4x)\big)$

$$= \ln(\operatorname{cosec} 4x) + \frac{x}{\operatorname{cosec} 4x}\times(-4\operatorname{cosec} 4x\cot 4x)$$

$$= -\ln(\sin 4x)(4x\cot 4x)$$

(d) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{\ln x}{2+3x}\right)$

$$= \frac{(2+3x)\frac{1}{x}-3(\ln x)}{(2+3x)^2}$$

$$= \frac{2+3x-3x\ln x}{x(2+3x)^2}$$

13
Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(\ln\left[(1-2x)(3+x)\right])$

$=\frac{\mathrm{d}}{\mathrm{d}x}(\ln(1-2x)+\ln(3+x))$

$=-\frac{2}{1-2x}+\frac{1}{3+x}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\ln\left(\frac{x+1}{3x-1}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\left(x+1\right)-\ln\left(3x-1\right)\right)$

$=\frac{1}{x+1}-\frac{3}{3x-1}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\sqrt{\frac{(x+1)^3}{x^2-1}}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{2}\ln\frac{(x+1)^3}{\sqrt{x^2-1}}\right)$

$=\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(3\ln(x+1)-\ln(x^2-1)\right)$

$=\frac{1}{2}\left(\frac{3}{x+1}-\frac{2x}{(x^2-1)}\right)$

$=\frac{1}{2}\left(\frac{3(x-1)}{(x+1)(x-1)}-\frac{2x}{(x^2-1)}\right)$

$=\frac{1}{2}\left(\frac{3x-3}{(x^2-1)}-\frac{2x}{(x^2-1)}\right)$

$=\frac{3x-1}{2(x^2-1)}$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln\frac{\sec x}{\sqrt{1-x^2}}\right)$

$=\frac{\mathrm{d}}{\mathrm{d}x}[\ln\sec x-\frac{1}{2}\ln(1-x^2)]$

$=\frac{\sec x\tan x}{\sec x}-\frac{1}{2}\left[\frac{-2x}{1-x^2}\right]$

$=\tan x+\frac{x}{1-x^2}$

Exercise 7

F Derivatives of Inverse Trigonometric Functions

14
Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}(\tan^{-1} 4x)$

$= \frac{1}{1+(4x)^2}\frac{\mathrm{d}}{\mathrm{d}x}(4x)$

$= \frac{1}{1+16x^2} \times (4)$

$= \frac{4}{1+16x^2}$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(\sin^{-1}(x^3+2x))$

$= \frac{3x^2+2}{\sqrt{1-(x^3+2x)^2}}$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}\left(\frac{1+x}{1-x}\right)\right)$

$= \left[\frac{1}{1+\left(\frac{1+x}{1-x}\right)^2}\right] \cdot \left[\frac{(1-x)\frac{\mathrm{d}}{\mathrm{d}x}(1+x)-(1+x)\frac{\mathrm{d}}{\mathrm{d}x}(1-x)}{(1-x)^2}\right]$

$= \left[\frac{(1-x)^2}{(1-x)^2+(1+x)^2}\right] \cdot \left[\frac{(1-x)+(1+x)}{(1-x)^2}\right]$

$= \left[\frac{2}{(1-x)^2+(1+x)^2}\right]$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin\left(\cos^{-1}(3x)\right)\right)$

$= \cos\left(\cos^{-1}(3x)\right) \times \frac{-3}{\sqrt{1-9x^2}}$

$= (3x) \times \frac{-3}{\sqrt{1-9x^2}}$

$= \frac{-9x}{\sqrt{1-9x^2}}$

(e) $\dfrac{d}{dx}\left(\dfrac{1}{\sin^{-1}3x}\right)^3$

$$= 3\left(\frac{1}{\sin^{-1}3x}\right)^2 \times \frac{d}{dx}\left[\sin^{-1}3x\right]^{-1}$$

$$= \frac{3}{(\sin^{-1}3x)^2} \times (-1)\left[\sin^{-1}3x\right]^{-2} \times \frac{1}{\sqrt{1-(3x)^2}}(3)$$

$$= -\frac{3}{(\sin^{-1}3x)^2} \times \frac{3}{(\sin^{-1}3x)^2\sqrt{1-9x^2}}$$

$$= -\frac{9}{\left(\sin^{-1}3x\right)^4\sqrt{1-9x^2}}$$

(f) $\dfrac{d}{dx}\sqrt{\cos^{-1}\left(\dfrac{x}{2}\right)}$

$$= \frac{1}{2}\left(\cos^{-1}\left(\frac{x}{2}\right)\right)^{-\frac{1}{2}} \frac{-1}{\sqrt{1-\left(\frac{x}{2}\right)^2}} \times \frac{1}{2}$$

$$= \frac{1}{4\sqrt{\cos^{-1}\left(\frac{x}{2}\right)}} \times \frac{-1}{\sqrt{1-\frac{x^2}{4}}}$$

$$= -\frac{1}{4\sqrt{\cos^{-1}\left(\frac{x}{2}\right)\left(1-\frac{x^2}{4}\right)}}$$

$$= -\frac{1}{2\sqrt{(4-x^2)\cos^{-1}\left(\frac{x}{2}\right)}}$$

15
Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(x\tan^{-1}x\right)$

$= x\frac{\mathrm{d}}{\mathrm{d}x}\left(\tan^{-1}x\right)+\left(\tan^{-1}x\right)\frac{\mathrm{d}}{\mathrm{d}x}x$

$= \frac{x}{1+x^2}+\tan^{-1}x$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left[\mathrm{e}^{2x}\cos^{-1}(3x)\right]$

$= \mathrm{e}^{2x}\left(\frac{-3}{\sqrt{1-(3x)^2}}\right)+2\mathrm{e}^{2x}\cos^{-1}(3x)$

$= \mathrm{e}^{2x}\left[2\cos^{-1}(3x)-\frac{3}{\sqrt{1-9x^2}}\right]$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x}\ \sin^{-1}\ \sqrt{x}\right)$

$= \mathrm{e}^{2x}\left(\frac{1}{\sqrt{1-x}}\right)\frac{1}{2}x^{-\frac{1}{2}}+\sin^{-1}\sqrt{x}\times(2\mathrm{e}^{2x})$

$= \frac{\mathrm{e}^{2x}}{2\sqrt{x}}\left(\frac{1}{\sqrt{1-x}}\right)+2\mathrm{e}^{2x}\sin^{-1}\sqrt{x}$

$= \frac{\mathrm{e}^{2x}}{2}\left[\frac{1}{\sqrt{x-x^2}}+4\sin^{-1}\sqrt{x}\right]$

(d) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{\tan^{-1}(2x)}\sec^2 x\right)$

$= \mathrm{e}^{\tan^{-1}(2x)}(2\sec x\sec x\tan x)+\sec^2 x\left(\mathrm{e}^{\tan^{-1}(2x)}\right)\left(\frac{1}{1+4x^2}(2)\right)$

$= \mathrm{e}^{\tan^{-1}(2x)}(2\sec^2 x\tan x)+\sec^2 x\left(\mathrm{e}^{\tan^{-1}(2x)}\right)\left(\frac{2}{1+4x^2}\right)$

$= 2\mathrm{e}^{\tan^{-1}(2x)}\sec^2 x\left[\tan x+\frac{1}{1+4x^2}\right]$

Exercise 7

G Differentiation of Implicit Functions

16
Solution

(a) Given $x^2 + y^3 = xy$

Differentiate implicitly with respect to x:

$$2x + 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = x\frac{\mathrm{d}y}{\mathrm{d}x} + y$$

$$x\frac{\mathrm{d}y}{\mathrm{d}x} - 3y^2 \frac{\mathrm{d}y}{\mathrm{d}x} = 2x - y$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}(x - 3y^2) = 2x - y$$

$$\therefore \ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{2x - y}{x - 3y^2}$$

(b) Given $x^3 y + xy^3 = 1$

$$\frac{\mathrm{d}}{\mathrm{d}x}\left(x^3 y + xy^3\right) = \frac{\mathrm{d}}{\mathrm{d}x}(1)$$

$$3x^2 y + x^3 \frac{\mathrm{d}y}{\mathrm{d}x} + y^3 + x(3y^2)\frac{\mathrm{d}y}{\mathrm{d}x} = 0$$

$$(x^3 + 3xy^2)\frac{\mathrm{d}y}{\mathrm{d}x} = -3x^2 y - y^3$$

$$\therefore \ \frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{3x^2 y + y^3}{x^3 + 3xy^2}$$

(c) Given $3x^2 - 7y^2 + 4xy - 8x = 0$

$$\frac{\mathrm{d}}{\mathrm{d}x}(3x^2) - \frac{\mathrm{d}}{\mathrm{d}x}(7y^2) + \frac{\mathrm{d}}{\mathrm{d}x}(4xy) - \frac{\mathrm{d}}{\mathrm{d}x}(8x) = 0$$

$$6x - 14y\frac{\mathrm{d}y}{\mathrm{d}x} + 4x\frac{\mathrm{d}y}{\mathrm{d}x} + 4y - 8 = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{6x + 4y - 8}{14y - 4x}$$

$$= \frac{3x + 2y - 4}{7y - 2x}$$

(d) Given $xy+\tan^{-1}y=x^2$

Differentiate implicitly with respect to x:

$$x\frac{\mathrm{d}y}{\mathrm{d}x}+y+\frac{1}{1+y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=2x$$

$$\left(x+\frac{1}{1+y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2x-y$$

$$\left(\frac{x+xy^2+1}{1+y^2}\right)\frac{\mathrm{d}y}{\mathrm{d}x}=2x-y$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{(2x-y)(1+y^2)}{1+xy^2+x}$$

(e) Given $\sin x+\cos y=xy$

$$\cos x-\sin y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)=y+x\frac{\mathrm{d}y}{\mathrm{d}x}$$

$$(x+\sin y)\frac{\mathrm{d}y}{\mathrm{d}x}=\cos x-y$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\frac{\cos x-y}{x+\sin y}$$

(f) Given $\frac{1}{x}-\frac{1}{y}=\frac{1}{a}$, where a is an arbitrary constant.

$$-\frac{1}{x^2}+\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=0$$

$$\frac{1}{y^2}\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{x^2}$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x}=\left(\frac{y}{x}\right)^2$$

17 Solution

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\ln(x^x) = \dfrac{\mathrm{d}}{\mathrm{d}x}\left(x\ln x\right)$

$$= x\left(\frac{1}{x}\right) + \ln x$$

(b) Given $\sqrt[x]{y} = \sqrt[y]{x}$

$$y^{\frac{1}{x}} = x^{\frac{1}{y}}$$

Taking logarithm on both sides

$$\ln y^{\frac{1}{x}} = \ln x^{\frac{1}{y}}$$

$$\frac{1}{x}\ln y = \frac{1}{y}\ln x$$

$$y\ln y = x\ln x$$

Differentiating both sides with respect to x

$$y\cdot\frac{1}{y}\cdot\frac{\mathrm{d}y}{\mathrm{d}x} + \frac{\mathrm{d}y}{\mathrm{d}x}\cdot\ln y = x\cdot\frac{1}{x} + 1\cdot\ln x$$

$$(1+\ln y)\frac{\mathrm{d}y}{\mathrm{d}x} = 1 + \ln x$$

$$\therefore\ \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1+\ln x}{1+\ln y}$$

18
Solution

Let $y^2+2xy+x^2-2e^x=7$ .... (1)
Differentiating (1) both sides with respect to x

$$2y\frac{dy}{dx}+\left(2x\frac{dy}{dx}+2y\right)+2x-2e^x=0$$

$$2y\frac{dy}{dx}+2x\frac{dy}{dx}=2e^x-2x-2y$$

$$\frac{dy}{dx}=\frac{e^x-x-y}{x+y}$$

Substituting $x=0$ into (1)

$$y^2+2(0)y+(0)^2-2e^0=7$$

$$y^2-2(1)=7$$

$$y^2=9$$

$$y=\pm3$$

Substituting $x=0$ and $y=3$ into $\frac{dy}{dx}=\frac{e^x-x-y}{x+y}$.

$$\frac{dy}{dx}=\frac{e^0-0-3}{0+3}$$

$$=-\frac{2}{3}$$

Substituting $x=0$ and $y=-3$ into $\frac{dy}{dx}=\frac{e^x-x-y}{x+y}$.

$$\frac{dy}{dx}=\frac{e^0-0-(-3)}{0+(-3)}$$

$$=-\frac{4}{3}$$

$\therefore$ the gradients of the tangents to the curve at the points A and B are $-\frac{2}{3}$ and $-\frac{4}{3}$ respectively.

19
Solution

(a) $(x+y)^2+2(x-y)^2=24$ .... (1)

Differentiating (1) both sides with respect to x

$$2(x+y)\left(1+\frac{dy}{dx}\right)+4(x-y)\left(1-\frac{dy}{dx}\right)=0$$

$$(x+y)+(x+y)\frac{dy}{dx}+(2x-2y)-(2x-2y)\frac{dy}{dx}=0$$

$$x+y+2x-2y=(2x-2y-x-y)\frac{dy}{dx}$$

$$3x-y=(x-3y)\frac{dy}{dx}$$

$$\therefore\ \frac{dy}{dx}=\frac{3x-y}{x-3y}\quad\text{(Shown)}$$

(b) For tangent parallel to x-axis, $\frac{dy}{dx}=0$.

i.e. $0=\frac{3x-y}{x-3y}$

$\therefore\ 3x-y=0$

$y=3x$ .... (2)

Substituting (2) into (1)

$$(x+3x)^2+2(x-3x)^2=24$$
$$16x^2+8x^2=24$$
$$x^2=1$$
$$x=1\text{ or }-1$$

Substituting $x=1$ into (2)

$y=3(1)$
$=3$

$\therefore$ the coordinates are (1, 3)

Substituting $x=-1$ into (2)

$y=3(-1)$
$=-3$

$\therefore$ the coordinates are (1, -3)

$\therefore$ the coordinates at which the tangent is parallel to the x-axis are $(1,3)$ and $(-1,-3)$.

For tangent parallel to y-axis, $\frac{dy}{dx}$ is undefined

i.e. $x-3y=0$

$x=3y$ .... (3)

Substituting (3) into (1)

$$(3y+y)^2+2(3y-y)^2=24$$
$$16y^2+8y^2=24$$
$$y^2=1$$
$$y=1 \text{ or } -1$$

Substituting $y=1$ into (3)

$$x=3(1)$$
$$=3$$

$\therefore$ the coordinates are (3, 1)

Substituting $y=-1$ into (3)

$$x=3(-1)$$
$$=-3$$

$\therefore$ the coordinates are $(-3,-1)$

$\therefore$ the coordinates at which the tangent is parallel to the y-axis are (3, 1) and $(-3,-1)$.

(c) When the curve cuts y-axis, i.e. $x=0$.

Substituting $x=0$ into (1)

$$y^2+2y^2=24$$
$$3y^2=24$$
$$y^2=8$$
$$y=\pm2\sqrt{2}$$

The coordinates at which the curve cuts the y-axis are $(0,-2\sqrt{2})$ and $(0,2\sqrt{2})$.

Substituting $x=0$ and $y=2\sqrt{2}$ into $\frac{dy}{dx}=\frac{3x-y}{x-3y}$.

$$\frac{dy}{dx}=\frac{3(0)-2\sqrt{2}}{0-3(2\sqrt{2})}=\frac{1}{3}$$

The gradient of the curve at $(0,2\sqrt{2})$ is $\frac{1}{3}$.

Substituting $x=0$ and $y=-2\sqrt{2}$ into $\frac{dy}{dx}=\frac{3x-y}{x-3y}$.

$$\frac{dy}{dx}=\frac{3(0)-(-2\sqrt{2})}{0-3(-2\sqrt{2})}=\frac{1}{3}$$

The gradient of the curve at $(0,-2\sqrt{2})$ is $\frac{1}{3}$.

When the curve cuts x-axis, i.e. $y = 0$.

Substituting $y = 0$ into (1)

$$x^2 + 2x^2 = 24$$

$$3x^2 = 24$$

$$x^2 = 8$$

$$\therefore \quad x = \pm 2\sqrt{2}$$

The curve cuts the x-axis at $(-2\sqrt{2}, 0)$ and $(2\sqrt{2}, 0)$.

Substituting $x = 2\sqrt{2}$ and $y = 0$ into $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x - y}{x - 3y}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(2\sqrt{2}) - (0)}{2\sqrt{2} - 3(0)} = 3$$

The gradient of the curve at $(2\sqrt{2}, 0)$ is 3.

Substituting $x = -2\sqrt{2}$ and $y = 0$ into $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3x - y}{x - 3y}$.

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3(-2\sqrt{2}) - (0)}{-2\sqrt{2} - 3(0)} = 3$$

The gradient of the curve at $(-2\sqrt{2}, 0)$ is 3.

20
Solution

Given $\quad \operatorname{cosec} y = x$

Differentiate both sides with respect to x

$$(-\operatorname{cosec} y \cot y)\frac{dy}{dx} = 1$$

$$\therefore \frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y} \quad ....\ (1)$$

Using Pythagorean Identity $\cot^2 y + 1 = \operatorname{cosec}^2 y$

$$\cot^2 y = \operatorname{cosec}^2 y - 1$$

$$\cot y = \pm\sqrt{\operatorname{cosec}^2 y - 1}$$

Given $0 < y < \frac{\pi}{2}$, $\tan y > 0$. Therefore $\cot y > 0$.

So, $\quad \cot y = \sqrt{(\operatorname{cosec} y)^2 - 1]}$ .... (2)

Substituting (2) into (1)

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y\sqrt{(\operatorname{cosec} y)^2 - 1}}$$

$$= -\frac{1}{x\sqrt{x^2 - 1}} \quad \text{(Deduced)}$$

Alternative Method

Given $\quad \operatorname{cosec} y = x$

Differentiate both sides with respect to x

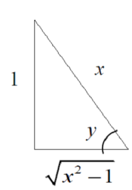
$$-\operatorname{cosec} y \cot y \frac{dy}{dx} = 1$$

$$\therefore \quad \frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y}$$

Since $\quad \operatorname{cosec} y = x$

$$\therefore \quad \sin y = \frac{1}{x}$$

By constructing the right angle triangle,



$$\tan y = \frac{1}{\sqrt{x^2 - 1}}$$

$$\frac{dy}{dx} = -\frac{1}{\operatorname{cosec} y \cot y}$$

$$= -\frac{\tan y}{\operatorname{cosec} y}$$

$$= -\frac{1}{x\sqrt{x^2 - 1}} \quad \text{(Deduced)}$$

21
Solution

Given $\tan^{-1}x+\tan^{-1}y+\tan^{-1}(xy)=\dfrac{7}{12}\pi$ .... (1)

Substituting $x=1$ into (1).

$$\tan^{-1}(1)+\tan^{-1}y+\tan^{-1}y=\frac{7}{12}\pi$$

$$\frac{\pi}{4}+2\tan^{-1}y=\frac{7\pi}{12}$$

$$\tan^{-1}y=\frac{\pi}{6}$$

$$y=\tan\frac{\pi}{6}$$

$$y=\frac{1}{\sqrt{3}}$$

$\therefore\ y=\dfrac{1}{\sqrt{3}}$ when $x=1$.

(a) $\dfrac{d}{dx}\left[\tan^{-1}(xy)\right]$

$$=\frac{1}{1+(xy)^2}\left(y+x\frac{dy}{dx}\right)$$

(b) Differentiate (1) both sides with respect to x

$$\frac{1}{1+x^2}+\left(\frac{1}{1+y^2}\right)\frac{dy}{dx}+\frac{y+x\frac{dy}{dx}}{1+(xy)^2}=0 \text{ .... (2)}$$

Substituting $x=1$ and $y=\dfrac{1}{\sqrt{3}}$ into (2)

$$\frac{1}{1+1^2}+\left(\frac{1}{1+\left(\frac{1}{\sqrt{3}}\right)^2}\right)\frac{dy}{dx}+\frac{\frac{1}{\sqrt{3}}+(1)\frac{dy}{dx}}{1+\left(1\times\frac{1}{\sqrt{3}}\right)^2}=0$$

$$\frac{1}{2}+\left(\frac{3}{4}\right)\frac{dy}{dx}+\frac{\frac{1}{\sqrt{3}}+\frac{dy}{dx}}{\frac{4}{3}}=0$$

$$\frac{1}{2}+\left(\frac{3}{4}\right)\frac{dy}{dx}+\frac{3}{4}\left(\frac{1}{\sqrt{3}}+\frac{dy}{dx}\right)=0$$

$$\frac{1}{2}+\left(\frac{3}{4}\right)\frac{dy}{dx}+\frac{3}{4\sqrt{3}}+\left(\frac{3}{4}\right)\frac{dy}{dx}=0$$

$$\left(\frac{6}{4}\right)\frac{dy}{dx}=-\frac{1}{2}-\frac{3}{4\sqrt{3}}$$

$$\frac{dy}{dx}=\frac{2}{3}\left(-\frac{1}{2}-\frac{3}{4\sqrt{3}}\right)$$

$\therefore\ \dfrac{dy}{dx}=-\dfrac{1}{3}-\dfrac{1}{2\sqrt{3}}$ (Shown)

Exercise 7

H Differentiation of Parametric Equations

22
Solution

(a) Given $x = (2t+1)^2$ .... (1)

and $y = 2t^3$ .... (2)

Differentiating (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 2(2t+1) \times 2 = 4(2t+1)$$

Differentiating (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 6t^2$$

Using the Chain Rule,

$$\begin{aligned} \frac{\mathrm{d}y}{\mathrm{d}x} &= \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}} \\ &= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}x}{\mathrm{d}t} \\ &= \frac{6t^2}{4(2t+1)} \\ &= \frac{3t^2}{2(2t+1)} \end{aligned}$$

$$\therefore \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3t^2}{2(2t+1)}$$

(b) Given $x = \dfrac{1}{1+t^2}$ .... (1)

and $y = \dfrac{t}{1+t^2}$ .... (2)

Differentiating (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = (2t)(1+t^2)^{-2}$$
$$= \frac{-2t}{(1+t^2)^2}$$

Differentiating (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{(1+t^2)(1)-(t)(2t)}{(1+t^2)^2}$$
$$= \frac{1-t^2}{(1+t^2)^2}$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\frac{\mathrm{d}y}{\mathrm{d}t}}{\frac{\mathrm{d}x}{\mathrm{d}t}}$$
$$= \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}x}{\mathrm{d}t}$$
$$= \frac{1-t^2}{(1+t^2)^2} \times \frac{(1+t^2)^2}{-2t}$$
$$= \frac{t^2-1}{2t}$$

$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{t^2-1}{2t}$

(c) Given $x = t + \mathrm{e}^t$ .... (1)

and $y = t + \mathrm{e}^{-t}$ .... (2)

Differentiating (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 + \mathrm{e}^t$$

Differentiating (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 1 - \mathrm{e}^{-t}$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$
$$= \frac{1-\mathrm{e}^{-t}}{1+\mathrm{e}^t}$$

$\therefore \dfrac{\mathrm{d}y}{\mathrm{d}x} = \dfrac{1-\mathrm{e}^{-t}}{1+\mathrm{e}^t}$

(d) Given $x = 2t + \sin 2t$ .... (1)

and $y = \cos 2t$ .... (2)

Differentiating (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{dt}} = 2 + 2\cos 2t$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{dt}} = -2\sin 2t$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$

$$= \frac{-2\sin 2t}{2 + 2\cos 2t}$$

$$= \frac{-\sin 2t}{1 + \cos 2t}$$

$$= \frac{-2\sin t \cos t}{1 + (2\cos^2 t - 1)}$$

$$= \frac{-2\sin t \cos t}{2\cos^2 t - 1)}$$

$$= -\tan t$$

$$\therefore \ \frac{\mathrm{d}y}{\mathrm{d}x} = -\tan t$$

23
Solution

Given $x = t - \frac{1}{t}$ .... (1)

and $y = t + \frac{1}{t}$ .... (2)

Differentiating (1) with respect to t

$$\frac{dx}{dt} = 1 + \frac{1}{t^2}$$
$$= \frac{t^2+1}{t^2}$$

Differentiate (2) with respect to t

$$\frac{dy}{dt} = 1 - \frac{1}{t^2}$$
$$= \frac{t^2-1}{t^2}$$

Using Chain Rule

$$\frac{dy}{dx} = \frac{\frac{t^2-1}{t^2}}{\frac{t^2+1}{t^2}} = \frac{t^2-1}{t^2+1}.$$

When the gradient is 0, i.e. $\frac{dy}{dx} = 0.$

$$\therefore \quad \frac{t^2-1}{t^2+1} = 0$$
$$t^2 - 1 = 0$$
$$t = 1 \quad \text{or} \quad -1$$

The values of t when the gradient is zero are 1 or -1.

24
Solution

Given $x = 2\cos^3 t$ .... (1)

and $y = \sin^3 t$ .... (2)

Differentiating (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 6\cos^2 t(-\sin t)$$
$$= -6\sin t\cos^2 t$$

Differentiate (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = 3\sin^2 t\cos t$$

Using Chain Rule

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{3\sin^2 t\cos t}{-6\sin t\cos^2 t}$$
$$= -\frac{1}{2}\tan t$$

When $t = \dfrac{\pi}{6}$,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = -\frac{1}{2}\tan\frac{\pi}{6}$$
$$= -\frac{\sqrt{3}}{6}$$

The exact value of $\dfrac{\mathrm{d}y}{\mathrm{d}x}$ when $t = \dfrac{\pi}{6}$ is $-\dfrac{\sqrt{3}}{6}$.

Exercise 7

I Derivatives of Higher Order Derivatives

25
Solution

(a) $\frac{dy}{dx} = 2e^{\sin x} \cos x$

$= y \cos x$

$\frac{d^2 y}{dx^2} = \frac{dy}{dx}(\cos x) - y \sin x \quad \triangleleft \text{ since } \frac{dy}{dx} = y\cos x, \ \therefore \ y = y\cos x$

$= \frac{dy}{dx}(\cos x) - \frac{1}{\cos x}\left(\frac{dy}{dx}\right)\sin x$

$= \frac{dy}{dx}(\cos x) - \frac{dy}{dx}\left(\frac{\sin x}{\cos x}\right)$

$\therefore \ \frac{d^2 y}{dx^2} = \frac{dy}{dx}(\cos x - \tan x)$ (Proved)

26
Solution

$y = \ln(1+\sin x)$

Differentiate both sides with respect to x

$\frac{dy}{dx} = \frac{\cos x}{1+\sin x}$

Differentiate both sides with respect to x

$\frac{d^2 y}{dx^2} = \frac{-\sin x(1+\sin x) - \cos x(\cos x)}{(1+\sin x)^2}$

$= \frac{-\sin x - 1}{(1+\sin x)^2}$

$= -\frac{1}{1+\sin x}$

Differentiate both sides with respect to x

$\frac{d^3 y}{dx^3} = \frac{1}{(1+\sin x)^2}(\cos x)$

$\frac{d^3 y}{dx^3} = -\left(\frac{\cos x}{1+\sin x}\right)\left(\frac{-1}{1+\sin x}\right)$

$\frac{d^3 y}{dx^3} = \left(\frac{dy}{dx}\right)\left(\frac{d^2 y}{dx^2}\right)$ (Shown)

27
Solution

Given $y = e^{\sin^{-1} x}$

Differentiate both sides with respect to x

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}} e^{\sin^{-1} x}$$

$$\sqrt{1-x^2}\frac{dy}{dx} = e^{\sin^{-1} x}$$

$$\sqrt{1-x^2}\frac{dy}{dx} = y \text{ .... } (*)$$

Differentiate both sides with respect to x

$$\frac{1}{2}\frac{-2x}{\sqrt{1-x^2}}\frac{dy}{dx} + \sqrt{1-x^2}\frac{d^2y}{dx^2} = \frac{dy}{dx}$$

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = \sqrt{1-x^2}\frac{dy}{dx} \qquad \triangleleft \sqrt{1-x^2}\frac{dy}{dx} = y \ \text{(from } (*)\text{)}$$

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = y$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0 \quad \text{(Shown)}$$

Alternative Method

Given $y = e^{\sin^{-1} x}$

$$\ln y = \sin^{-1} x$$

Differentiate both sides with respect to x

$$\frac{1}{y}\frac{dy}{dx} = \frac{1}{\sqrt{1-x^2}}$$

$$\sqrt{1-x^2}\frac{dy}{dx} = y \text{ .... } (*)$$

Differentiate both sides with respect to x

$$\frac{1}{2}\frac{-2x}{\sqrt{1-x^2}}\frac{dy}{dx} + \sqrt{1-x^2}\frac{d^2y}{dx^2} = \frac{dy}{dx}$$

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = \sqrt{1-x^2}\frac{dy}{dx}$$

$$-x\frac{dy}{dx} + (1-x^2)\frac{d^2y}{dx^2} = y \qquad \triangleleft \sqrt{1-x^2}\frac{dy}{dx} = y \ \text{(from } (*)\text{)}$$

$$(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} - y = 0 \quad \text{(Shown)}$$

28
Solution

Given $y = \sin\left[\ln(1+2x)\right]$

Differentiating with respect to x

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \cos\left[\ln(1+2x)\right].\frac{2}{1+2x}$$

$$(1+2x)\frac{\mathrm{d}y}{\mathrm{d}x} = 2\cos\left[\ln(1+2x)\right]$$

Differentiating with respect to x

$$(1+2x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\frac{\mathrm{d}y}{\mathrm{d}x} = -2\sin\left[\ln(1+2x)\right].\frac{2}{1+2x}$$

$$(1+2x)^2\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2(1+2x)\frac{\mathrm{d}y}{\mathrm{d}x} + 4y = 0 \quad \text{(Shown)}, \quad \text{where } k = 4.$$

29
Solution

Given $y = \sqrt{3+\sin x+\cos x}$

$$y^2 = 3+\sin x+\cos x$$

$$2y\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = \cos x - \sin x$$

$$2y\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2}\right) + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = -\sin x - \cos x$$

$$= 3 - y^2$$

$$\therefore\ 2y\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + 2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 + y^2 = 3 \quad \text{(Shown)}$$

Exercise 7

J Exam Style Questions

30
Solution

(a) $\dfrac{\mathrm{d}}{\mathrm{d}x}\left(\dfrac{ax}{\sqrt{x^3+1}}\right)$

$$=\frac{\sqrt{x^3+1}\times a-ax\times\left(\frac{1}{2}\right)\left(x^3+1\right)^{-\frac{1}{2}}\left(3x^2\right)}{\left(\sqrt{x^3+1}\right)^2}$$

$$=\frac{a\left(x^3+1\right)^{\frac{1}{2}}-\frac{3}{2}ax^3\left(x^3+1\right)^{-\frac{1}{2}}}{x^3+1}$$

$$=\frac{a\left(x^3+1\right)^{\frac{1}{2}}}{x^3+1}-\frac{-\frac{3}{2}ax^3\left(x^3+1\right)^{-\frac{1}{2}}}{x^3+1}$$

$$=a\left(x^3+1\right)^{-\frac{1}{2}}-\frac{3}{2}ax^3\left(x^3+1\right)^{-\frac{3}{2}}$$

(b) $\dfrac{\mathrm{d}}{\mathrm{d}x}(2x^2-x-6)^{-4}$

$$=-4(2x^2-x-6)^{-5}(4x-1)$$

$$=-\frac{4(4x-1)}{(2x^2-x-6)^5} \qquad \text{(Shown)}$$

(c) $y=\left(\dfrac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}\times\frac{(1-\sin x)\cos x-(1+\sin x)(-\cos x)}{(1-\sin x)^2}$$

$$\frac{\mathrm{d}y}{\mathrm{d}x}=\frac{1}{2}\left(\frac{1+\sin x}{1-\sin x}\right)^{\frac{1}{2}}\times\frac{2\cos x}{(1-\sin x)^2}$$

$$=\left(\frac{1-\sin x}{1+\sin x}\right)^{\frac{1}{2}}\frac{\cos x}{(1-\sin x)^2}$$

$$=\frac{1}{(1+\sin x)^{\frac{1}{2}}}\times\frac{\cos x}{(1-\sin x)^{-\frac{1}{2}}(1-\sin x)^2}$$

$$=\frac{\cos x}{(1+\sin x)^{\frac{1}{2}}(1-\sin x)^{\frac{1}{2}}(1-\sin x)}$$

$$=\frac{\cos x}{(1-\sin^2 x)^{\frac{1}{2}}(1-\sin x)}$$

$$=\frac{\cos x}{(\cos^2 x)^{\frac{1}{2}}(1-\sin x)}$$

$$=\frac{\cos x}{\cos x(1-\sin x)}$$

$$=\frac{1}{1-\sin x}$$

31
Solution

(a) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\sin^{-1}(\sqrt{x})\right)$

$$= \frac{1}{\sqrt{1-(\sqrt{x})^2}} \frac{\mathrm{d}}{\mathrm{d}x}(\sqrt{x})$$

$$= \frac{1}{\sqrt{1-x}} \ \frac{1}{2\sqrt{x}}$$

$$= \frac{1}{2\sqrt{x(1-x)}}$$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\log_2 x\right) = \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\ln x}{\ln 2}\right)$

$$= (\ln 2)\frac{\mathrm{d}}{\mathrm{d}x}\left((\ln x)\right)$$

$$= (\ln 2)\times\frac{1}{x}$$

$$= \frac{\ln 2}{\ln x}$$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x}\tan x\right)$

$$= 2\mathrm{e}^{2x}\tan x + \mathrm{e}^{2x}\sec^2 x$$

$$= \mathrm{e}^{2x}\left(2\tan x + \sec^2 x\right)$$

32
Solution

(a) $\dfrac{d}{dx}\left[x^2\ln\left(\sec 2x\right)\right]$

$= x^2\dfrac{1}{\sec 2x}\left(\sec 2x\tan 2x\right)(2)+\ln\left(\sec 2x\right)(2x)$

$= 2x^2\tan 2x+2x\ln\left(\sec 2x\right)$

$= 2x(x\tan 2x+\ln(\sec 2x))$

(b) $\dfrac{d}{dx}\tan^{-1}\sqrt{e^{2x}-1}$

$= \dfrac{1}{1+\left(\sqrt{e^{2x}-1}\right)^2}\dfrac{1}{2}\left(e^{2x}-1\right)^{-\frac{1}{2}}e^{2x}(2)$

$= \dfrac{1}{1+e^{2x}-1}\left(e^{2x}-1\right)^{-\frac{1}{2}}e^{2x}$

$= \dfrac{1}{e^{2x}}\left(e^{2x}-1\right)^{-\frac{1}{2}}e^{2x}$

$= \dfrac{1}{\sqrt{e^{2x}-1}}$

(c) Let $y=x^{2x}$

Taking logarithm on both sides :

$\ln y=\ln x^{2x}$

$\ln y=2x\ln x$

Differentiating implicitly w.r.t. x on both sides :

$\dfrac{1}{y}\dfrac{dy}{dx}=(2)\ln x+2x\left(\dfrac{1}{x}\right)$

$\dfrac{1}{y}\dfrac{dy}{dx}=2\ln x+2$

$\dfrac{dy}{dx}=y(2\ln x+2)$

$= 2x^{2x}(\ln x+1)$

Alternative method

$$\frac{\mathrm{d}}{\mathrm{d}x}(x^{2x})$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left((\mathrm{e}^{\ln x})^{2x}\right)$$

$$= \frac{\mathrm{d}}{\mathrm{d}x}\left(\mathrm{e}^{2x\ln x}\right)$$

$$= \mathrm{e}^{2x\ln x}\frac{\mathrm{d}}{\mathrm{d}x}(2x\ln x)$$

$$= \mathrm{e}^{2x\ln x}\left((2)\ln x + 2x\left(\frac{1}{x}\right)\right)$$

$$= \mathrm{e}^{2x\ln x}(2\ln x + 2)$$

$$= 2x^{2x}(\ln x + 1)$$

33
Solution

(a) Let $y = \cos^{-1}\sqrt{1-x^2}$

$$\frac{dy}{dx} = -\frac{1}{\sqrt{1-(\sqrt{1-x^2})^2}} \times \frac{1}{2}(1-x^2)^{-\frac{1}{2}} \times (-2x)$$

$$= -\frac{1}{\sqrt{x^2}} \times \frac{1}{2\sqrt{1-x^2}} \times (-2x)$$

Since $0 < x < 1$

$$= \frac{1}{x}\left(\frac{1}{\sqrt{1-x^2}}\right)(x)$$

$$= \frac{1}{\sqrt{1-x^2}}$$

(b) $\frac{d}{dx}\left[\operatorname{cosec}\left(\ln(x^2+3)\right)\right]$

$$= -\operatorname{cosec}\left(\ln(x^2+3)\right)\cot\left(\ln(x^2+3)\right)\cdot\left(\frac{1}{x^2+3}(2x)\right)$$

$$= -\frac{2x\operatorname{cosec}\left(\ln(x^2+3)\right)\cot\left(\ln(x^2+3)\right)}{x^2+3}$$

Alternative Method

$$\frac{d}{dx}\left[\operatorname{cosec}\left(\ln(x^2+3)\right)\right]$$

$$= \frac{d}{dx}\left[\frac{1}{\sin\left(\ln(x^2+3)\right)}\right]$$

$$= \frac{d}{dx}\left[\sin\left(\ln(x^2+3)\right)\right]^{-1}$$

$$= -\left[\sin\left(\ln(x^2+3)\right)\right]^{-2} \times \left[\cos\left(\ln(x^2+3)\right)\cdot\left(\frac{1}{x^2+3}(2x)\right)\right]$$

$$= -\frac{1}{\left[\sin\left(\ln(x^2+3)\right)\right]^2} \times \left[\frac{2x\cos\left(\ln(x^2+3)\right)}{x^2+3}\right]$$

$$= -\frac{2x\cos\left(\ln(x^2+3)\right)}{(x^2+3)\sin^2\left(\ln(x^2+3)\right)}$$

(c) $\frac{\mathrm{d}}{\mathrm{d}x}\left(\ln x\right)^{20}$

$= \frac{1}{x}(20)\left(\ln x\right)^{19}$

$= \frac{20}{x}\left(\ln x\right)^{19}$

34
Solution

$$x = e^{3t}\cos 3t$$

$$\frac{dx}{dt} = 3e^{3t}\cos 3t + e^{3t}(-3\sin 3t)$$

$$= 3e^{3t}(\cos 3t - \sin 3t)$$

$$y = e^{3t}\sin 3t$$

$$\frac{dy}{dt} = 3e^{3t}\sin 3t + e^{3t}(3\cos 3t)$$

$$= 3e^{3t}(\sin 3t + \cos 3t)$$

Using the Chain Rule

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}}$$

$$= \frac{dy}{dt} \times \frac{dx}{dt}$$

$$= \frac{3e^{3t}(\sin 3t + \cos 3t)}{3e^{3t}(\cos 3t - \sin 3t)}$$

$$= \frac{(\sin 3t + \cos 3t)}{(\cos 3t - \sin 3t)}$$

$$= \frac{\sqrt{2}\sin\left(3t + \frac{\pi}{4}\right)}{\sqrt{2}\cos\left(3t + \frac{\pi}{4}\right)}$$

$$= \tan\left(3t + \frac{\pi}{4}\right) \quad \text{(Shown)}$$

35
Solution

Given $x = t - \sin t$ .... (1)
and $y = 1 - \cos t$ .... (2)

Differentiating (1) with respect to t

$$\frac{\mathrm{d}x}{\mathrm{d}t} = 1 - \cos t$$

Differentiating (2) with respect to t

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \sin t$$

Using the Chain Rule,

$$\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{\mathrm{d}y}{\mathrm{d}t} \times \frac{\mathrm{d}t}{\mathrm{d}x}$$

$$= \sin t \times \frac{1}{1 - \cos t}$$

$$= \frac{2\sin\frac{t}{2}\cos\frac{t}{2}}{1 - \left(1 - 2\sin^2\frac{t}{2}\right)}$$

$$= \cot\frac{t}{2}$$

$\therefore\ k = \frac{1}{2}$

36
Solution

$$x = \frac{1}{2}(e^{3t} + 2e^{-3t})$$

$$\frac{dx}{dt} = \frac{1}{2}(3e^{3t} - 6e^{-3t})$$

$$= \frac{3}{2}(e^{3t} - 2e^{-3t})$$

$$y = \frac{1}{2}(e^{3t} - 2e^{-3t})$$

$$\frac{dy}{dt} = \frac{1}{2}(3e^{3t} + 6e^{-3t})$$

$$= \frac{3}{2}(e^{3t} + 2e^{-3t})$$

Using the Chain Rule,

$$\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$$

$$= \frac{\frac{3}{2}(e^{3t} + 2e^{-3t})}{\frac{3}{2}(e^{3t} - 2e^{-3t})}$$

$$= \frac{e^{3t} + 2e^{-3t}}{e^{3t} - 2e^{-3t}}$$

When $t = \frac{1}{3}\ln 2$,

$$\frac{dy}{dx} = \frac{e^{3\left(\frac{1}{3}\ln 2\right)} + 2e^{-3\left(\frac{1}{3}\ln 2\right)}}{e^{3\left(\frac{1}{3}\ln 2\right)} - 2e^{-3\left(\frac{1}{3}\ln 2\right)}}$$

$$\frac{dy}{dx} = \frac{e^{\ln 2} + 2e^{-\ln 2}}{e^{\ln 2} - 2e^{-\ln 2}}$$

$$\frac{dy}{dx} = \frac{e^{\ln 2} + 2e^{\ln 2^{-1}}}{e^{\ln 2} - 2e^{\ln 2^{-1}}}$$

$$\frac{dy}{dx} = \frac{2 + 2(2^{-1})}{2 - 2(2^{-1})}$$

$$\frac{dy}{dx} = 3$$

Gradient of tangent is 3.

$\therefore$ gradient of normal to the curve at the point where $t = \frac{1}{3}\ln 2$ is $-\frac{1}{3}$.

(b) Given $x = \dfrac{1}{2}(e^{3t} + 2e^{-3t})$ $\triangleleft$ square both sides

$$x^2 = \left[\frac{1}{2}(e^{3t} + 2e^{-3t})\right]^2$$

$$x^2 = \frac{1}{4}(e^{6t} + 4 + 4e^{-6t}) \text{ .... (1)}$$

and $y = \dfrac{1}{2}(e^{3t} - 2e^{-3t})$ $\triangleleft$ square both sides

$$y^2 = \left[\frac{1}{2}(e^{3t} - 2e^{-3t})\right]^2$$

$$y^2 = \frac{1}{4}(e^{6t} - 4 + 4e^{-6t}) \text{ .... (2)}$$

Taking $(1) - (2)$

$$x^2 - y^2 = \frac{1}{4}(e^{6t} + 4 + 4e^{-6t}) - \frac{1}{4}(e^{6t} - 4 + 4e^{-6t})$$

$$x^2 - y^2 = 2$$

The cartesian equation of the curve is $x^2 - y^2 = 2$.

Restriction on the values of x:

$y^2 \geq 0$, for real values of y

$y^2 + 2 \geq 2$ $\triangleleft$ add 2 both sides

$x^2 \geq 2$ $\triangleleft$ $x^2 - y^2 = 2 \Rightarrow x^2 = 2 + y^2$

$x^2 - 2 \geq 0$

$(x+\sqrt{2})(x+\sqrt{2}) \geq 0$

$\therefore$ $x \leq -\sqrt{2}$ or $x \geq \sqrt{2}$

Since $e^{3t} > 0$ and $2e^{-3t} > 0$, $x > 0$

The restriction on the values of x is $x \geq \sqrt{2}$.

37
Solution

(a) $x^2+2xy-\ln(e^2+2y)=3$

Differentiating both sides with respect to x

$$2x+2x\frac{dy}{dx}+2y-\frac{2\frac{dy}{dx}}{e^2+2y}=0$$

$$x+x\frac{dy}{dx}+y-\frac{\frac{dy}{dx}}{e^2+2y}=0$$

$$\frac{\frac{dy}{dx}}{e^2+2y}-x\frac{dy}{dx}=x+y$$

$$\frac{dy}{dx}-x(e^2+2y)\frac{dy}{dx}=(x+y)(e^2+2y)$$

$$\frac{dy}{dx}(1-e^2x-2xy)=(x+y)(e^2+2y)$$

$$\therefore\ \frac{dy}{dx}=\frac{(x+y)(e^2+2y)}{1-xe^2-2xy}$$

(b) Let $y=\cos^{-1}x$

$$\cos y=x$$

$$-\sin y\frac{dy}{dx}=1$$

$$\frac{dy}{dx}=-\frac{1}{\sin y}$$

For $0\le y\le\pi$, $\sin y\ge 0$.

$$\therefore\ \sin y=\sqrt{1-\cos^2 y}$$

$$=\sqrt{1-x^2}$$

$$\frac{dy}{dx}=-\frac{1}{\sqrt{1-x^2}},\quad \text{for } -1<x<1$$

38
Solution

(a) Given $y = x\tan^{-1} x$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = x\left(\frac{1}{1+x^2}\right) + \tan^{-1} x \qquad \triangleleft \text{ apply product rule on RHS}$$

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = x + (1+x^2)\tan^{-1} x \qquad \triangleleft \text{ multiply } x \text{ to all sides}$$

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + (1+x^2)x\tan^{-1} x \qquad \triangleleft \text{ replace } y = x\tan^{-1} x$$

$$(1+x^2)\frac{\mathrm{d}y}{\mathrm{d}x} = x^2 + (1+x^2)y \quad \text{(Proved)}$$

(b) Given $\sqrt{x} + \sqrt{y} = \sqrt{a}$

$$x^{\frac{1}{2}} + y^{\frac{1}{2}} = \sqrt{a}$$

Differentiating both sides with respect to x

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0$$

$$\sqrt{y} + \sqrt{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = 0$$

$$\sqrt{x}\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right) = -\sqrt{y} \qquad \triangleleft \text{ square both sides}$$

$$x\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = y$$

Differentiating both sides with respect to x

$$x2\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 = \frac{\mathrm{d}y}{\mathrm{d}x}$$

$$\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)\left(2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 1\right) = 0$$

$$\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \text{or} \quad 2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 1 = 0$$

$\frac{\mathrm{d}y}{\mathrm{d}x} = 0$ (reject as $\frac{\mathrm{d}y}{\mathrm{d}x} < 0$, since $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-\sqrt{y}}{\sqrt{x}} < 0$)

Hence, $2x\frac{\mathrm{d}^2y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x} - 1 = 0$ (Shown)

39
Solution

Given
$$y = \frac{\cos x}{1-2x}$$

$$(1-2x)y = \cos x$$

$$\frac{\mathrm{d}}{\mathrm{d}x}[(1-2x)y] = \frac{\mathrm{d}}{\mathrm{d}x}(\cos x)$$

$$(1-2x)\frac{\mathrm{d}y}{\mathrm{d}x} + y(-2) = -\sin x$$

$$(1-2x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + \frac{\mathrm{d}y}{\mathrm{d}x}(-2) - 2\frac{\mathrm{d}y}{\mathrm{d}x} = -\cos x$$

$$(1-2x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} = -\cos x$$

$$(1-2x)\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} - 4\frac{\mathrm{d}y}{\mathrm{d}x} = -(1-2x)y$$

$$(1-2x)\left(\frac{\mathrm{d}^2 y}{\mathrm{d}x^2} + y\right) - 4\frac{\mathrm{d}y}{\mathrm{d}x} = 0 \quad \text{(Shown)}$$

40
Solution

(a) Given $y = \dfrac{1}{\sqrt{2e^x - 1}}$

$(2e^x - 1)y^2 = 1$ .... (1)

Differentiating (1) with respect to x

$2(2e^x - 1)y\dfrac{dy}{dx} + 2e^x y^2 = 0$ ◁ replace $(2e^x - 1) = \dfrac{1}{y^2}$ from (1)

$2y\left(\dfrac{1}{y^2}\right)\dfrac{dy}{dx} + 2e^x y^2 = 0$

$y\left(\dfrac{1}{y^2}\right)\dfrac{dy}{dx} + e^x y^2 = 0$

$\dfrac{1}{y^3}\dfrac{dy}{dx} = -e^x y^2$

$\therefore \dfrac{dy}{dx} = -y^3 e^x$ .... (2) (Shown)

Alternative Method

$y = \dfrac{1}{\sqrt{2e^x - 1}}$

$y = (2e^x - 1)^{-\frac{1}{2}}$

$\dfrac{dy}{dx} = -\dfrac{1}{2}(2e^x - 1)^{-\frac{3}{2}}(2e^x)$

$\dfrac{dy}{dx} = -\left[(2e^x - 1)^{-\frac{1}{2}}\right]^3 e^x$

$\dfrac{dy}{dx} = -y^3 e^x$ .... (1)

(b) Differentiating (2) with respect to x

$\dfrac{d^2y}{dx^2} = -y^3 e^x - 3y^2\dfrac{dy}{dx}e^x$ ◁ $\dfrac{dy}{dx} = -y^3 e^x$

$\dfrac{d^2y}{dx^2} = \dfrac{dy}{dx} - 3y^2\dfrac{dy}{dx}e^x$

$y\dfrac{d^2y}{dx^2} = y\dfrac{dy}{dx} + 3\dfrac{dy}{dx}(-y^3 e^x)$ ◁ $\dfrac{dy}{dx} = -y^3 e^x$

$y\dfrac{d^2y}{dx^2} = y\dfrac{dy}{dx} + 3\dfrac{dy}{dx}\left(\dfrac{dy}{dx}\right)$

$y\dfrac{d^2y}{dx^2} = y\dfrac{dy}{dx} + 3\left(\dfrac{dy}{dx}\right)^2$ (shown)